Lesson 33, Double Integrals, part I.

I. Geometry

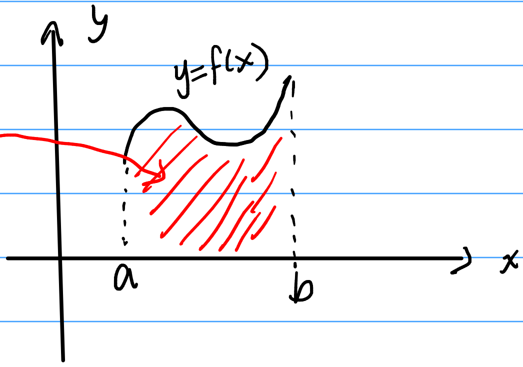
II. Computing double integrals

Announcements.

1. Quiz Friday lessons 33 and 34.

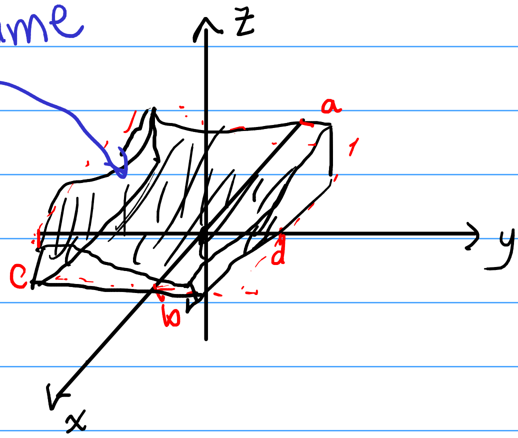
I. Geometry

Rcl: $\int_a^b f(x)\,dx$ Area



What if we have a function of 2 variables.

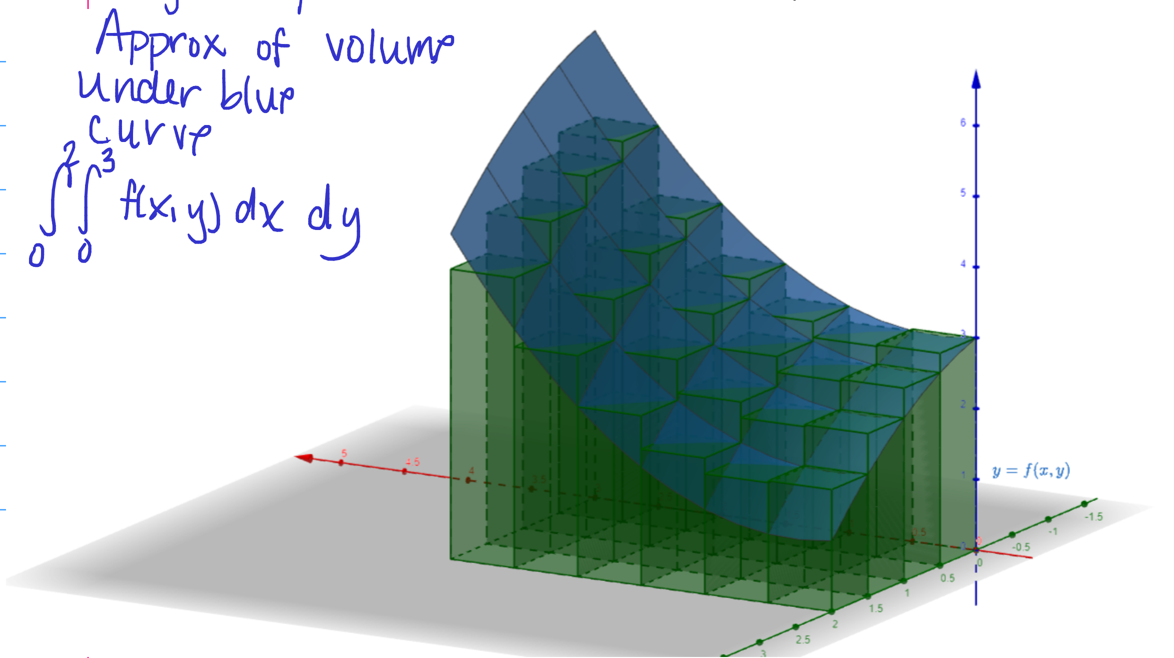
$\int_a^b \int_c^d f(x,y)\,dx\,dy$ Volume



Geogebra pic.

Approx of volume under blue curve

$\int_0^2 \int_0^3 f(x,y)\,dx\,dy$



II. Computing Double integrals

Similar to partial derivatives
When integrating with respect to (wrt) the variable w, you pretend that all other variables are constants

$$\int_a^b \underbrace{\int_c^d f(x,y)\, dx}_{\text{Start on the inside}}\, dy$$

$\boxed{\text{Ex 1}}$ (Rowgawski et. al. § 15.1 #19)

$$\int_1^3 \underbrace{\int_0^2 x^3 y\, dy}\, dx = \int_1^3 \left[x^3 \frac{y^2}{2} \Big|_{y=0}^{y=2} \right] dx$$

$$= \int_1^3 \left[x^3 \cdot \frac{4}{2} - x^3 \cdot \frac{0}{2} \right] dx = \int_1^3 2x^3\, dx = \frac{2x^4}{4} \Big|_{x=1}^{x=3}$$

$$\frac{1}{2}(3)^4 - \frac{1}{2}(1)^4 = \frac{81}{2} - \frac{1}{2} = \frac{80}{2} = 40$$

$\boxed{\text{Ex 2}}$ (Rowgawski et. al §15.1 #26)

$$\int_2^6 \underbrace{\int_1^4 y^2\, dx}\, dy = \int_2^6 \left[y^2 x \Big|_{x=1}^{x=4} \right] dy$$

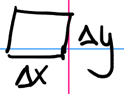
$$= \int_2^6 \left[y^2 \cdot 4 - y^2 \cdot 1 \right] dy = \int_2^6 3y^2\, dy = y^3 \Big|_{y=2}^{y=6} = 6^3 - 2^3$$

$$= 216 - 8$$

$$= 208$$

Notation: $\iint_R f(x,y)\, dA$ — dA stands for $dy\,dx$ or $dx\,dy$

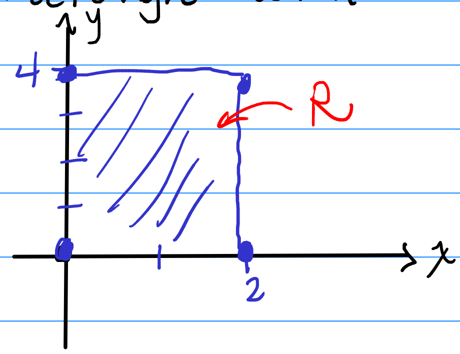
$dy\,dx$ or $dx\,dy$



R: Region of integration.
So far this has been a rectangle — but not always!

Ex 3 (Rog. et. al §15.1 #40)

Evaluate $\iint_R \frac{y}{x+1}\, dA$ where R is the rectangle with vertices $(0,0), (2,0), (0,4)$, and $(2,4)$



2 options.

$$\int_0^4 \underbrace{\int_0^2 \frac{y}{x+1}\, dx}\, dy \quad \text{OR} \quad \int_0^2 \underbrace{\int_0^4 \frac{y}{x+1}\, dy}\, dx$$

<u>Option 1:</u>

$$\int_0^4 \underbrace{\int_0^2 \frac{y}{x+1}\, dx}\, dy = \int_0^4 \left[y \ln|x+1| \Big|_{x=0}^{x=2} \right] dy$$

$$= \int_0^4 \left[y\ln(3) - \underbrace{y\ln(1)}_{=0} \right] dy$$

$$= \int_0^4 y\ln(3)\, dy = \frac{y^2}{2}\cdot \ln(3) \Big|_{y=0}^{y=4} = 8\ln(3) - 0$$

$$= 8\ln(3)$$

Not all regions are rectangular.

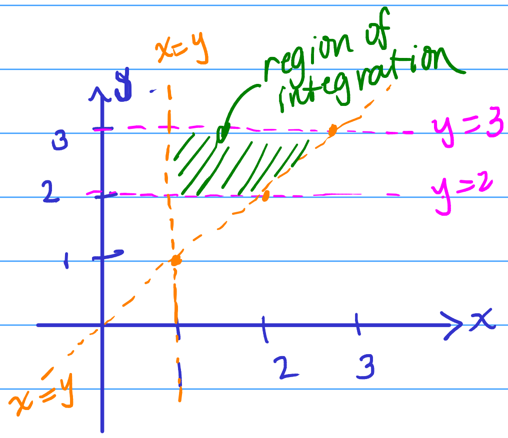
$\boxed{\text{Ex}}$ $\displaystyle\int_2^3\int_1^y (15yx^2+1)\,dx\,dy$

outside integral limits are #'s

What is our region of integration?

$y: 2 \to 3$ $\quad\quad$ $y=2 \to y=3$

$x: 1 \to y$ $\quad\quad$ $x=1 \to \boxed{x=y}$ line.



$$\underbrace{\int_2^3\int_1^y (15yx^2+1)\,dx}\,dy = \int_2^3 \left[\frac{15yx^3}{3} + x \Big|_{x=1}^{x=y}\right] dy$$

$5yx^3 + x$

$$= \int_2^3 \left[(5y\cdot y^3 + y) - (5y\cdot 1^3 + 1)\right] dy$$

$$= \int_2^3 \left[5y^4 + y - 5y - 1\right] dy$$

$$= \int_2^3 \left[5y^4 - 4y - 1\right] dy$$

$$= y^5 - 2y^2 - y \Big|_2^3$$

$$= (3^5 - 2(9) - 3) - (2^5 - 2(4) - 2) = 200$$

You try it:

Evaluate $\displaystyle\int_0^2\int_0^4 \frac{y}{x+1}\,dy\,dx$ and verify that it equals $8\ln(3)$. (See Ex 3)